STABILITY OF A VISCOELASTIC BODY IN THE FLOW OF AN IDEAL FLUID

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When a fluid flows around a plated elastic body an interesting flow regime occurs when undamped oscillations are excited and (or) maintained in the material making up the plating due to the energy of the flow field. The study of hydroelastic oscillations of this kind is important in the computation of the strength of materials and also in the problem of reducing the resistance of the body to the flow. The latter problem has been studied extensively, mainly experimentally. In a series of theoretical papers (see [1, 2], for example) the direction of study taking into account the interaction of the vortex structures of the flow with the surface waves of the streamlined elastic body appears to be the most promising.

Modern computational methods and techniques allow one to solve, in principle, the problem of hydroelastic oscillations in the complete (nonlinear) formulation. However this formulation leads to complicated and expensive numerical calculations of the amplitude and energy parameters. The linear formulation of the problem can be treated analytically up to obtaining of the so-called "critical parameters" of the problem. For example, the critical velocity of an incompressible fluid flowing around an elastic half-space was considered in [3].

In the present paper we analyze, in the linear formulation, the behavior of the critical parameters of the flow, taking into account the finite thickness of the elastic layer, the compressibility of the fluid, its density and other parameters, such as internal damping (friction) of the layer.

1. We consider a viscoelastic layer of thickness h in the xz plane ($-h \leqslant z \leqslant 0$, $-\infty < x < \infty$). The surface layer z = 0 interacts with a two-dimensional potential flow of an ideal compressible fluid.

The stresses and displacements in the viscoelastic body are related by the equations [4]

$$\begin{gathered}\sigma_{xz}=\mu\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)+\frac{\partial}{\partial t}\eta\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right),\\ \sigma_{xx}=\left(K+\frac{4}{3}\mu\right)\frac{\partial u_x}{\partial x}+\left(K-\frac{2}{3}\mu\right)\frac{\partial u_z}{\partial z}+\frac{\partial}{\partial t}\left[\left(\zeta+\frac{4}{3}\eta\right)\frac{\partial u_x}{\partial x}+\left(\zeta-\frac{2}{3}\eta\right)\frac{\partial u_z}{\partial z}\right],\\ \sigma_{zz}=\left(K+\frac{4}{3}\mu\right)\frac{\partial u_z}{\partial z}+\left(K-\frac{2}{3}\mu\right)\frac{\partial u_x}{\partial x}+\frac{\partial}{\partial t}\left[\left(\zeta+\frac{4}{3}\eta\right)\frac{\partial u_z}{\partial z}+\left(\zeta-\frac{2}{3}\eta\right)\frac{\partial u_x}{\partial x}\right],\end{gathered} \tag{1.1}$$

where K is the compressibility; μ is the shear modulus; ζ, η are the viscosities of the material (small quantities); u_x, u_z are the displacements along the x and z axes; t is the time.

On the upper boundary of the layer we have

$$\sigma_{xz}=0,\ \sigma_{zz}=-p_0 \tag{1.2}$$

(p_0 is the pressure of the flow field). On the rigidly fixed bottom of the layer we have

$$u_x=0,\ u_z=0. \tag{1.3}$$

Introducing the friction coefficients

$$\varepsilon_l=\sqrt{(\zeta+4\eta/3)/\rho},\ \varepsilon_t=\sqrt{\eta/\rho}, \tag{1.4}$$

which have the same forms as the well-known expressions for the velocities of compressional and shear waves [4]

$$c_l=\sqrt{(K+4\mu/3)/\rho},\ c_t=\sqrt{\mu/\rho} \tag{1.5}$$

(ρ is the density of the material), we can obtain equations for the displacement vectors corresponding to compressional and shear deformations:

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$$\frac{\partial^2 v}{\partial t^2} = c_l^2 \Delta v + \varepsilon_l^2 \frac{\partial}{\partial t} \Delta v, \quad \frac{\partial^2 u}{\partial t^2} = c_t^2 \Delta u + \varepsilon_t^2 \frac{\partial}{\partial t} \Delta u; \tag{1.6}$$

$$\operatorname{div} u = 0, \ \operatorname{rot} u = 0. \tag{1.7}$$

The components v_1, v_2, u_1, u_2 of these vectors are related to the "physical" displacements by the relations $u_x = u_1 + v_1$, $u_z = u_2 + v_2$.

We have the following well-known equation for the velocity potential φ of plane flow of an ideal compressible fluid

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} = \frac{1}{c_0^2}\left(\frac{\partial^2 \varphi}{\partial t^2} + 2U \frac{\partial^2 \varphi}{\partial x \partial t} + U^2 \frac{\partial^2 \varphi}{\partial x^2}\right), \tag{1.8}$$

where c_0 is the speed of sound in the fluid; U is the velocity of the unperturbed flow (infinitely far from the elastic layer).

The damped condition of the velocity perturbation (at $z = \infty$) will be

$$\partial \varphi / \partial x = U, \tag{1.9}$$

and the condition that the fluid cannot pass through the oscillating boundary $z = w(x, t)$ is

$$\partial u_z / \partial t + U \partial u_z / \partial x = \partial \varphi / \partial z, \tag{1.10}$$

which can be applied on the boundary $z = 0$, in view of the smallness of w.

The pressure in the fluid is found from the Cauchy–Lagrange integral

$$\frac{p_0}{\rho_0} + \frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla \varphi)^2 = \text{const} \tag{1.11}$$

(ρ_0 is the density of the fluid).

2. We limit ourselves to solutions periodic in x. Let λ be the wavelength, where $\lambda = 2\pi/k$, k is the wave number and ω is the frequency. The potential φ can be represented in the form

$$\varphi(x, z, t) = Ux + C\exp(-i\theta - K_1 z) \tag{2.1}$$

(C is an undetermined constant, $\theta = kx - \omega t$). With $K_1 = k\sqrt{1 - M_1}$, $M_1 = |U_{ph} - U|/c_0$, $U_{ph} = \omega/k$, (2.1) satisfies (1.8). The condition (1.10) can be satisfied by proper choice of the undetermined constant.

The displacement in the elastic layer is written as a wave of the same form

$$u_j = U_j(z)e^{-i\theta}, \ v_j = V_j(z)e^{-i\theta} \ (j = 1, 2). \tag{2.2}$$

Equations (1.6) and (1.7) then form a system of homogeneous linear ordinary differential equations for U_j and V_j.

The pressure in the fluid [after linearization of (1.11) and proper choice of the constant on the right-hand side] is $p_0 = -i\rho_0\omega C(U_{ph} - U)\exp(-i\theta)$. Then the boundary conditions (1.2) and (1.3) also transform into linear homogeneous relations.

We therefore obtain the following closed homogeneous system

$$U_j'' = \varkappa_t^2 U_j, \quad V_j'' = \varkappa_l^2 V_j \quad (j = 1, 2); \tag{2.3}$$

$$U_2' = ikU_1, \quad V_1' = -ikV_2; \tag{2.4}$$

$$U_1'(0) + V_1'(0) - ik(U_2(0) + V_2(0)) = 0; \tag{2.5}$$

$$(c_l^2 + \varepsilon_l^2)(U_2'(0) + V_2'(0)) - ik(c_l^2 - 2c_t^2 + i\omega(\varepsilon_l^2 - 2\varepsilon_t^2))(U_1(0) + V_1(0)) = iCk\frac{\rho_0}{\rho}(U_{ph} - U), \quad iC = \frac{U_{ph} - U}{\sqrt{1 - M_1^2}}(U_2(0) + V_2(0)); \tag{2.6}$$

$$U_j(-h) + V_j(-h) = 0 \ (j = 1, 2). \tag{2.7}$$

Here $\varkappa_t^2 = k^2(1 - \xi^2 D_t)$; $\varkappa_l^2 = k^2(1 - \delta^2\xi^2 D_l)$; $\varepsilon_u = \varepsilon_t^2/c_t$; $\varepsilon_v = \varepsilon_l^2/c_l$; $D_t = 1/(1 + ik\xi\varepsilon_u)$; $D_l = 1/(1 + ik\xi\delta\varepsilon_v)$; $\xi = \omega/(kc_t)$ is the dimensionless frequency.

The condition that there exist a nontrivial solution of the system (2.3) through (2.7) gives, after several reductions,

$$F(\xi) \equiv \alpha D_t (\xi - \gamma)^2 - \sqrt{1 - \mu^2(\xi - \gamma)^2}\Lambda = 0, \tag{2.8}$$

whose solution gives the frequency ξ and therefore also ω of the Rayleigh waves as a function of the various parameters of the problem. In (2.8) $\alpha = \rho_0/\rho$ is the dimensionless density of the fluid; $\gamma = U/c_t$ is the dimensionless velocity of the flow; $\delta = c_t/c_l$; $\mu = c_t/c_0$ is the compressibility index of the fluid; $\sigma = kh/2\pi = h/\lambda$ is the reduced (to the wavelength) thickness of the elastic layer; $\Lambda = (R_1S_1 - R_4S_2)/(R_1S_3 - R_4S_4)$, where $R_1 = B_1(Q + Q_{21}) - B_2Q_{11}$; $R_4 = B_1Q_{22} + B_2(Q - Q_{12})$;

$$\begin{gathered} S_1 = -A_1Q_{22} + A_2(Q + Q_{12});\ S_2 = A_1(Q - Q_{21}) + A_2Q_{11}; \\ S_3 = Q_{22} + x_l(Q - Q_{12});\ S_4 = Q + Q_{21} - x_lQ_{11}; \\ B_1 = 1 + x_t^2;\ B_2 = 2x_l;\ A_1 = 2x_t;\ A_2 = B_1; \\ Q_{11} = -2x_tE_tE_l;\ Q_{12} = -(1 + x_tx_l)E_l^2; \\ Q_{21} = -(1 + x_tx_l)E_t^2;\ Q_{22} = -2x_lE_tE_l; \\ Q = 1 - x_tx_l;\ E_t = \exp(-2\pi\sigma x_t);\ E_l = \exp(-2\pi\sigma x_l); \\ x_t = \sqrt{1 - \xi^2D_t},\ x_l = \sqrt{1 - \xi^2\delta^2D_l}. \end{gathered} \tag{2.9}$$

The general solution of (2.8) is a complicated quantity of the form $\xi(\gamma, \alpha, \delta, \mu, \sigma, \varepsilon_u, \varepsilon_v) = X + iY$. The parameters in this expression determine the stable region ($Y \geqslant 0$) and the unstable region ($Y < 0$) of the oscillations of the surface of the elastic layer. The parameters defining the boundary between these two regions are called the critical parameters. We will mainly study the dependence of the critical velocity γ_* on the other parameters.

3. We consider an elastic half-space without viscosity and with the condition $c_l \gg c_t$. Then $\varepsilon_u = \varepsilon_v = 0$, $\sigma = \infty$, $\delta = 0$. Equation (2.8) simplifies to

$$(2 - \xi^2)^2 - 4\sqrt{1 - \xi^2} + \frac{\alpha\xi^2(\xi - \gamma)^2}{\sqrt{1 - \mu^2(\xi - \gamma)^2}} = 0. \tag{3.1}$$

In the absence of the flow, i.e., when $\alpha = 0$, (3.1) has the known [4] roots $\xi_{1,2} = 0.955...$, which are the dimensionless natural frequencies of oscillation of the half-space (Rayleigh surface waves). If $\alpha \neq 0$ but $\mu = 0$ (an incompressible fluid) (3.1) reduces to an algebraic equation of the sixth degree. Varying γ continuously in the interval $0 \leqslant \gamma \leqslant \gamma_*$, one can follow the "trajectories" of the roots $\xi_{1,2}$ to the point where they join at $\gamma = \gamma_*$.

Detailed calculations and analysis of this kind have been given in [3]. However, this type of calculation becomes practically impossible for finite values of σ and also for $\mu \neq 0$, $\delta \neq 0$ because the form of equation (2.8) becomes very complicated. Therefore in the present paper we consider first the quantity $\gamma(\xi)$ rather than $\xi(\gamma)$. It follows from (2.8) that for real ξ in the interval $-1 \leq \xi \leq 1$

$$\gamma = \xi \pm \sqrt{-\frac{\mu^2\Lambda^2}{2\alpha^2} + \sqrt{\left(\frac{\mu^2\Lambda^2}{2\alpha^2}\right)^2 + \left(\frac{\Lambda}{\alpha}\right)^2}}. \tag{3.2}$$

From this general solution we have two interesting special cases:

1) rarefied compressible flow (air) around a layer of elastic plate, when $\alpha \sim 0.001$, $\mu \sim 0.1$, $\Lambda \sim 100$, and therefore $(\mu\Lambda/\alpha)^2 \gg \Lambda/\alpha$:

$$\gamma = \xi \pm 1/\mu; \tag{3.3}$$

2) flow of an incompressible fluid (water) around a thick rubber layer, when $\alpha \sim 1$, $\mu \sim 0$, $\Lambda = 1\text{-}10$:

$$\gamma = \xi \pm \sqrt{\Lambda/\alpha}. \tag{3.4}$$

The function $\gamma(\xi)$ for $|\xi| \leqslant 1$ is illustrated in Fig. 1, where $\alpha = 1$, $\mu = 0$, $\delta = 0$, $\sigma = 10$. The curve has the form of an inclined ellipse tangent to the two vertical lines $\xi = \xi_{1,2}$ at the points where they intersect the line $\gamma = \xi$. When $\sigma = \infty$ the calculation gives $\xi_{1,2} = 0.955...$, but when σ is small and finite (this quantity depends on the other parameters) $\xi_{1,2}$ lie outside of the interval $|\xi| \leqslant 1$. The upper extremum point in Fig. 1 corresponds to γ_*. The function $\gamma(\xi)$ is shown in Fig. 2 for $\alpha = 0.001$, $\sigma = \infty$ and $\mu = 0$, 0.01, 0.05, 0.10 (curves 1-5, respectively).

Analysis of (3.2)-(3.4), Fig. 2, and Table 1 (the dependence of γ_* on the thickness of the layer and the density of the fluid for $\mu = 0$, $\delta = 0$) shows that the critical

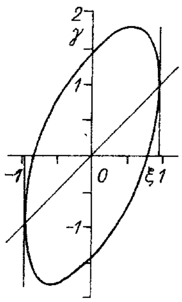

Fig. 1

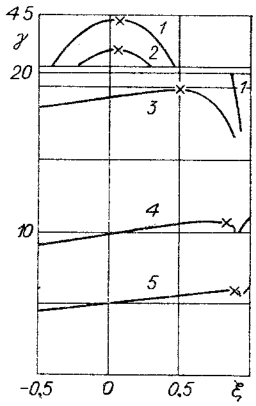

Fig. 2

TABLE 1

σ	γ_*			
	α			
	0,001	0,005	0,010	1,000
10,0	44,736	20,033	14,189	1,793
1,0	44,751	20,040	14,194	1,795
0,5	46,993	21,047	14,911	1,940
0,1	143,62	64,29	45,52	—
0,05	337,54	151,03	106,86	—

velocity increases with decreasing σ and α and decreases with increasing μ, which is in complete correspondence with the physical interpretation of the phenomenon.

4. When internal friction is taken into account (i.e., $\varepsilon_u \neq 0$, $\varepsilon_v \neq 0$) the function $F(\xi, \gamma)$ from (2.8) will be complex. The solution of this equation must be found by iteration (here the method of sections is applied) in the complex plane $\xi(\gamma) = X + iY$. An initial guess is chosen near the critical values of ξ, γ calculated without taking friction into account, and calculations were performed in the subcritical and supercritical regions.

Table 2 presents the dependence of X and Y on γ for different ε_u, $\varepsilon_v = \varepsilon_u$ ($\alpha = 1$, $\mu = 0$, $\delta = 0$, $\sigma = 5$, $k = \pi$). For these values of the parameters the value of γ_* calculated without taking into account friction is $\gamma_* = 1.79$.

We observe the following paradox: there exists a zone (for $\gamma < 2$) where the instability of of the elastic layer in the flow increases when the friction $k\varepsilon_u$ increases. Calculations for other values of the parameters ($\alpha = 0.001$, $\mu = 0.1$, $\delta = 0$, $\sigma = 0.5$, $\varepsilon_u = 0.05$, $\varepsilon_v = \varepsilon_u$) presented in Table 3 show that this is not an accidental occurrence, but a manifestation of Siegler's paradox, which is an old, but little-known, paradox, and is described in [5, 6], for example. As shown in [5], in the presence of nonconservative forces "the addition of dissipative forces can be a destabilizing influence."

The mechanism of this destabilization is shown in Fig. 3, where the "trajectories" of the complex roots ξ are plotted against γ. Trajectory a corresponds to the absence of friction. When $\gamma = \gamma_* = 1.79$ the real roots become equal, and for $\gamma > \gamma_*$ they are complex conjugates, which corresponds to instability ($Y < 0$). Trajectory b is obtained with friction taken into account ($\varepsilon_u = 0.05$, $\varepsilon_v = 0.1167$, $k = 1$, $\mu = 0$, $\alpha = 1$, $\delta = 0$). It is close to

TABLE 2

γ	X				−Y			
	ε_u				ε_u			
	0	0,5	0,1	0,02	0	0,5	0,1	0,2
1,75	0,455	0,410	0,358	0,279	0	0,094	0,135	0,167
1,80	0,638	0,480	0,410	0,316	0,067	0,140	0,172	0,198
1,85	0,633	0,536	0,457	0,353	0,186	0,194	0,213	0,213
1,90	0,688	0,580	0,499	0,387	0,256	0,248	0,255	0,263
1,95	0,714	0,617	0,537	0,420	0,312	0,298	0,296	0,296
2,00	0,740	0,650	0,571	0,452	0,359	0,343	0,336	0,329
2,05	0,767	0,681	0,604	0,482	0,402	0,385	0,375	0,362
2,10	0,794	0,710	0,635	0,512	0,442	0,425	0,412	0,394

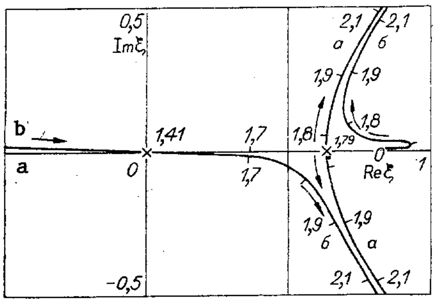

Fig. 3

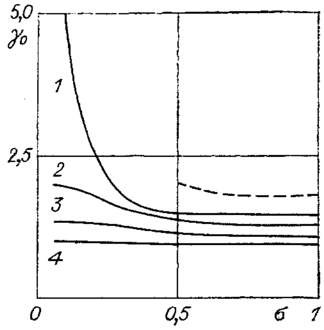

Fig. 4

TABLE 3

γ	X			−Y		
	k			k		
	0	1	3,14	0	1	3,14
10,6	0,618	0,618	0,617	0	0,002	0,005
10,7	0,723	0,723	0,721	0	0,003	0,008
10,8	0,834	0,833	0,826	0	0,006	0,015
10,9	0,980	0,956	0,929	0	0,027	0,033
11,0	1,048	1,035	1,013	0,096	0,079	0,056
11,1	1,899	1,086	1,079	0,119	0,099	0,066
11,2	1,126	1,128	1,136	0,120	0,098	0,056

trajectory a, but crosses over into the lower complex half-plane at a significantly smaller value of γ. The values of γ are given on the curves. A qualitatively similar plot was given in [5] for a viscoelastic model system.

In our case it is noteworthy that with friction taken into account curve b must pass through the point $\xi = 0$. At this value the effect of friction is absent, as follows from (2.9). Therefore the critical velocity $\gamma = \gamma_0$ will be $\gamma(0)$ regardless of the dependence on ε_u, ε_v. This value can be calculated from (3.2) after a complicated evaluation of an undefined term of the type 0/0 in the expression for $\Lambda(\xi)$ in the limit $\xi \to 0$.

Figure 4 shows the dependence of γ_0 on σ for $\mu = 0, 0.5, 0.75, 1.00$ (curves 1-4) for $\alpha = 1$, $\delta = 0$, $k = 1$, $\varepsilon_u = \varepsilon_v = 0.05$. For comparison, the dashed curve shows $\gamma_*(\sigma)$ calculated without taking into account friction for the case corresponding to curve 1.

For an infinitely thick layer a comparison can be carried out in Fig. 2, where γ_0 corresponds to the points of the curves lying on the axis $\xi = 0$, and γ_* corresponds to the extremum points marked by the crosses.

Therefore we see that in the formulation of the problem considered here it is possible to effectively and comparatively easily carry out a broad parameteric study of the critical parameters and the frequencies of the free oscillations of the elastic plating in the flow field. The calculations demonstrate the extremely significant effect of the thickness of the layer and the compressibility and density of the fluid on the critical velocity. The noted above destabilization with friction taken into account (the stability is significantly lowered) is a special case of phenomena characteristic of nonconservative stability problems.

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